Comment about quasi-isotropic solution of Einstein equations near cosmological singularity

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Abstract

We generalize for the case of arbitrary hydrodynamical matter the quasi-isotropic solution of Einstein equations near cosmological singularity, found by Lifshitz and Khalatnikov in 1960 for the case of the radiation-dominated Universe. It is shown that this solution always exists, but dependence of terms in the quasi-isotropic expansion acquires a more complicated form.

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In the paper [1] by Lifshitz and Khalatnikov, the quasi-isotropic solution of the Einstein equations near a cosmological singularity was found provided the Universe was filled by radiation with the equation of state $p = \frac{\varepsilon}{3}$. The metric of this solution was written down in the synchronous system of reference

$$ds^2 = dt^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta, \tag{1}$$

where spatial metric $\gamma_{\alpha\beta}$ near the singularity has the form

$$\gamma_{\alpha\beta} = t a_{\alpha\beta} + t^2 b_{\alpha\beta} + \cdots \tag{2}$$

where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are functions of spatial coordinates. The functions $a_{\alpha\beta}$ are chosen arbitrary, and then the functions $b_{\alpha\beta}$ and also the energy and velocity distributions for matter can be expressed through these functions (for details see [1] and also [2, 3]).

In correspondence with the standard cosmological model of the hot Universe, it was supposed that the natural equation of state for the matter near the cosmological singularity is that of radiation: $p = \varepsilon/3$. However, nowadays the situation has changed in connection with the development of inflationary cosmological models, which as an important ingredient contain inflaton scalar field or/and other exotic types of matter [4]. One can add also that the appearance of brane and M theory cosmological models [5] and the discovery of the cosmic acceleration [6] suggests that the matter playing essential role on different stages of cosmological evolution can obey very different equations of state [7]. Thus, generalization of the old quasi-isotropic solution of the Einstein equations near the cosmological singularity can be useful in this new context.

In this note we make such a generalization for the equation of state:

$$p = k\varepsilon, \tag{3}$$

where p denotes pressure and ε denotes energy density [1]. The Friedmann isotropic solution near the singularity for such a matter behaves as

$$a \sim a_0 t^m, \tag{4}$$

where

$$m = \frac{4}{3(1+k)}. \tag{5}$$

We look for an expression for a spatial metric in the following form:

$$\gamma_{\alpha\beta} = t^m a_{\alpha\beta} + t^n b_{\alpha\beta}, \tag{6}$$

where the power index m is given by Eq. (5). We leave the power index n free for some time, requiring only that

$$n > m. \tag{7}$$

[1] Actually, the authors have known this generalization for a long time but have never published it in detail in regular journals.

The inverse metric reads

$$\gamma^{\alpha\beta} = \frac{a^{\alpha\beta}}{t^m} - \frac{b^{\alpha\beta}}{t^{2m-n}}, \tag{8}$$

where $a^{\alpha\beta}$ is defined by the relation

$$a^{\alpha\beta} a_{\beta\gamma} = \delta^\alpha_\gamma \tag{9}$$

while the indices of all the other matrices are lowered and raised by $a_{\alpha\beta}$ and $a^{\alpha\beta}$, for example,

$$b^\alpha_\beta = a^{\alpha\gamma} b_{\gamma\beta}. \tag{10}$$

Let us write down also expressions for the extrinsic curvature, its contractions and its derivatives:

$$\kappa_{\alpha\beta} \equiv \frac{\partial\gamma_{\alpha\beta}}{\partial t} = m t^{m-1} a_{\alpha\beta} + n t^{n-1} b_{\alpha\beta}, \tag{11}$$

$$\kappa^\beta_\alpha = \frac{m\delta^\beta_\alpha}{t} + \frac{(n-m) b^\beta_\alpha}{t^{m-n+1}}, \tag{12}$$

$$\kappa^\alpha_\alpha = \frac{3m}{t} + \frac{(n-m) b}{t^{m-n+1}}, \tag{13}$$

$$\frac{\partial\kappa^\beta_\alpha}{\partial t} = -\frac{m\delta^\beta_\alpha}{t^2} - \frac{(m-n+1)(n-m) b^\beta_\alpha}{t^{m-n+2}}, \tag{14}$$

$$\frac{\partial\kappa^\alpha_\alpha}{\partial t} = -\frac{3m}{t^2} - \frac{(m-n+1)(n-m) b}{t^{m-n+2}}, \tag{15}$$

$$\kappa^\beta_\alpha \kappa^\alpha_\beta = \frac{3m^2}{t^2} + \frac{2m(n-m) b}{t^{m-n+2}}. \tag{16}$$

We need also an explicit expression for the determinant of the spatial metric:

$$\gamma \equiv \det \gamma_{\alpha\beta} = t^{3m}(1 + t^{n-m} b) \det a, \tag{17}$$

$$\dot\gamma \equiv \frac{\partial\gamma}{\partial t} = (3m t^{3m-1} + b(2m+n) t^{2m+n-1}) \det a, \tag{18}$$

$$\frac{\dot\gamma}{\gamma} = \frac{3m}{t}\left(1 + \frac{b(n-m) t^{n-m}}{3m}\right). \tag{19}$$

Now, using well-known expressions for the components of the Ricci tensor [3]:

$$R^0_0 = -\frac{1}{2}\frac{\partial\kappa^\alpha_\alpha}{\partial t} - \frac{1}{4}\kappa^\beta_\alpha \kappa^\alpha_\beta, \tag{20}$$

$$R^0_\alpha = \frac{1}{2}(\kappa^\beta_{\alpha;\beta} - \kappa^\beta_{\beta;\alpha}), \tag{21}$$

$$R^\beta_\alpha = -P^\beta_\alpha - \frac{1}{2}\frac{\partial \kappa^\beta_\alpha}{\partial t} - \frac{\dot{\gamma}}{4\gamma}\kappa^\beta_\alpha, \tag{22}$$

where P^β_α is a three-dimensional part of the Ricci tensor, and substituting into Eqs. (20)-(22) the expressions (11)-(19), one get

$$R^0_0 = \frac{3m(2-m)}{4t^2} - \frac{(n-1)(n-m)b}{2t^{m-n+2}}, \tag{23}$$

$$R^0_\alpha = \frac{n-m}{2t^{m-n+1}}(b^\beta_{\alpha;\beta} - b_{;\alpha}), \tag{24}$$

$$\begin{aligned} R^\beta_\alpha = & -\frac{\tilde{P}^\beta_\alpha}{t^m} + \frac{m(2-3m)\delta^\beta_\alpha}{4t^2} \\ & + \frac{(n-m)(2-2n-m)b^\beta_\alpha}{4t^{m-n+2}} - \frac{m(n-m)b\delta^\beta_\alpha}{4t^{m-n+2}}. \end{aligned} \tag{25}$$

Notice, that in Eq. (25) $\tilde{P}^\beta_\alpha$ denotes a three-dimensional Ricci tensor constructed by using the metrics $a_{\alpha\beta}$. The terms in the curvature tensor P^β_α, which are proportional to $\beta_{\alpha\beta}$ have the time dependence $\sim \frac{1}{t^{2m-n}}$ and are less divergent than the first term in the right-hand side of Eq. (25) provided the condition (7) is satisfied.

Now, let us write down the expressions for the components of the energy-momentum tensor of the perfect fluid

$$T_{ik} = (\varepsilon + p)u_i u_k - p g_{ik}, \tag{26}$$

satisfying the equation of state (3). Up to higher-order corrections, they have the following form :

$$T^0_0 = \varepsilon \tag{27}$$

$$T^0_\alpha = \varepsilon(k+1)u_\alpha, \tag{28}$$

$$T^\beta_\alpha = -k\varepsilon\delta^\beta_\alpha, \tag{29}$$

$$T = T^i_i = \varepsilon(1-3k). \tag{30}$$

Using the Einstein equations

$$R^j_i = 8\pi G(T^j_i - \frac{1}{2}\delta^j_i T), \tag{31}$$

one has from 00-component of these equations:

$$8\pi G\varepsilon = \frac{1}{3k+1}\left(\frac{3m(2-m)}{2t^2} - \frac{(n-1)(n-m)b}{t^{m-n+2}}\right), \tag{32}$$

and from 0α-component of these equations one has

$$u_\alpha = \frac{(n-m)(3k+1)(b^\beta_{\alpha;\beta} - b_{;\alpha})t^{n+1-m}}{3m(2-m)(k+1)}. \tag{33}$$

Now, writing down the spatial components of the Einstein equations, using the expressions (29)-(30) and the expression (32) for the energy density ε, one get:

$$\begin{aligned} &-\frac{\tilde{P}^\beta_\alpha}{t^m} + \frac{m(2-3m)\delta^\beta_\alpha}{4t^2} + \frac{(n-m)(2-2n-m)b^\beta_\alpha}{4t^{m-n+2}} \\ &-\frac{m(n-m)b\delta^\beta_\alpha}{4t^{m-n+2}} = \frac{(k-1)\delta^\beta_\alpha}{3k+1}\left(\frac{3m(2-m)}{2t^2} - \frac{(n-1)(n-m)b}{t^{m-n+2}}\right). \end{aligned} \tag{34}$$

Using the relation (5) it is easy to check that the terms proportional to $\frac{1}{t^2}$ in the left- and right-hand sides of Eq. (34) cancel each other. On the other hand, the only way to cancel the term $\frac{\tilde{P}^\beta_\alpha}{t^m}$ is to require that the terms proportional to $\frac{1}{t^{m-n+2}}$ behave as the term $\frac{1}{t^m}$, i.e.

$$n = 2. \tag{35}$$

In this case, the condition of cancellation of terms proportional to $\frac{1}{t^m}$ gives the following expression for the tensor b^β_α:

$$b^\beta_\alpha = \frac{4\tilde{P}^\beta_\alpha}{m^2-4} + \frac{\tilde{P}\delta^\beta_\alpha(-3m^2+12m-4)}{3m(m-3)(m^2-4)}. \tag{36}$$

Using the relation (5) one can rewrite the expression (36) in the following form:

$$b^\beta_\alpha = -\frac{9(k+1)^2}{(3k+5)(3k+1)}\left(\tilde{P}^\beta_\alpha + \frac{(3k^2-6k-5)\delta^\beta_\alpha\tilde{P}}{9k+5}\right). \tag{37}$$

It is easy to see that the Eq. (37) expressing the second-order correction for the spatial metric (2) is well defined for all the types of hydrodynamical matter with $0 \le k \le 1$, including the stiff matter, i.e. a fluid with the equation of state $p = \varepsilon$.

Now, using the relation

$$\tilde{P}_{\alpha}^{\beta;\alpha} = \frac{1}{2}\tilde{P}_{;\beta}, \tag{38}$$

and the formulae (37) and (33) we arrive to the following expression for the three-dimensional velocity u_α:

$$u_\alpha = -\frac{27k(k+1)^3\tilde{P}_{;\alpha}}{8(3k+5)(9k+5)}t^{3-\frac{4}{3(k+1)}}. \tag{39}$$

Note that the velocity flow is potential. Actually, it can be shown that this important property remains in all higher orders of perturbative expansion for the quasi-isotropic solution. Similarly, the expression for the energy density of matter (32) can be rewritten as

$$8\pi G\varepsilon = \frac{4}{3(k+1)^2t^2} + \frac{9(k+1)\tilde{P}}{2(9k+5)t^{\frac{4}{3(k+1)}}}. \tag{40}$$

It is straightforward to check that further terms of perturbative expansion (2) for the metric $\gamma_{\alpha\beta}$ have the following form:

$$\gamma_{\alpha\beta} = t^m a_{\alpha\beta} + t^2 b_{\alpha\beta} + t^{2+(2-m)}c_{\alpha\beta} + \cdots. \tag{41}$$

Thus, we have seen that this expansion has a curious feature. The order of its first term t^m is defined by the equation of state of the matter (3), the second order term always has the behavior $\sim t^2$, while logarithmic distance between orders is equal to $2-m$.

It is easy to understand that the quasi-isotropic expansion does work if and only if the first term in the right-hand side of Eq. (6) is smaller than the second one. Remembering that $n = 2$ for any value of m and using the equation (5), we get the following restriction on the parameter k from the equation of state (3):

$$k > -\frac{1}{3}. \tag{42}$$

Thus, for the values $k \leq -1/3$ the quasi-isotropic expansion at small times may not be constructed. It is interesting to notice, however, that for $k = const < -\frac{1}{3}$, a quasi-isotropic-like solution arises as a late-time ($t \to \infty$) attractor for generic inhomogeneous evolution of space-time [8] (this regime is the power-law inflation [9] actually). Of course, perturbative expansion is

made in inverse powers of t in that case. Also, we would like to note that different aspects of relation between the quasi-isotropic expansion and other approximation schemes were considered in detail in the papers [10].

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